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Analysis of Milne's Device for the Finite Correction Mode of the Adams PC Methods II

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1. Introduction

In the paper [1], the author discussed an accurete method for estimating local truncation errors and, as its application, an accurate method for estimating global truncation errors. In that paper, he mentioned two theorems on the behavior of the difference between the values of the predictor and of the corrector for the Adams PC methods both in the $P(EC)^mE$ mode and in the $P(EC)^m$ mode. The proofs, however, were not given there. Recently, the author gave the proof of the theorm in the $P(EC)^m$ mode [2][3]. The purpose of this paper is to give the proof of the theorem in the $P(EC)^mE$ mode.

2. Preliminaries

We consider the initial value problem of the differential equation

$$y' = f(x,y), \quad y(a) = y_0 \quad (a \le x \le b), \tag{1}$$

where we denote by $y(x)$ the solution of this problem. The step points are given by

$$x_n = a + nh \quad (n = 0, 1, ..., N), \qquad h = (b-a)/N\,,$$

where N is the total number of the steps. Let p be the order of the Adams PC methods. Put

$$v = n + p - 1\,.$$

In what follows, we assume that $f(x,y)$ *in* Eq.(1) *is suficiently smooth on the regions in question.* We assume that *the solution* $y(x)$ *of* Eq.(1) *exists.* Let y_μ $(\mu = 0, 1, \ldots, p-1)$ are p starting values and let

$$e_\mu = y_\mu - y(x_\mu) \quad (\mu = 0, 1, \ldots, p-1).$$

We also assume that y_μ*'s are chosen so that*

$$e_\mu = O(h^q) \quad (q \geq p+1; \mu = 0, 1, \ldots, p-1).$$

Let

$$g(x) = f_y(x, y(x)), \quad g_v^{(k)} = g^{(k)}(x_v), \quad g_v = g(x_v).$$

The formulae of the Adams predictor–corrector method of order p in the $P(EC)^m E$ mode are given as follows:

(2) $$y_v^{[0]} = y_{v-1}^{[m]} + h\sum_{j=1}^{p} a_{pj} f_{v-j}^{[m]} = y_{v-1}^{[m]} + h\sum_{j=0}^{p-1} \gamma_j \nabla^j f_{v-1}^{[m]},$$

(3) $$\begin{aligned} y_v^{[m]} &= y_{v-1}^{[m]} + hb_{p0} f_v^{[m-1]} + h\sum_{j=1}^{p-1} b_{pj} f_{v-j}^{[m]} \\ &= y_{v-1}^{[m]} + h\sum_{j=0}^{p-1} \hat{\gamma}_j \nabla^j f_v^{[m]} + hb_{p0}(f_v^{[m-1]} - f_v^{[m]}), \end{aligned}$$

where $y_k^{[i]}$ is the i-th correction of $y_k^{[0]}$, $f_k^{[i]} = f(x_k, y_k^{[i]})$,∇ is the backward difference operator,

$$\gamma_j = \int_0^1 s(s+1)\cdots(s+j-1)ds/j!,$$

$$\hat{\gamma}_j = \int_0^1 (s-1)s\cdots(s+j-2)ds/j!.$$

For the formulae Eqs.(2) and (3), let us define the local truncation errors at x_n by

$$T_{p1n} = T_{p1}(x_v, y(x_v); h) = y(x_v) - y(x_v - h) - h\sum_{j=1}^{p} a_{pj} f(x_v - jh, y(x_v - jh)),$$

and

$$T_{p2n} = T_{p2}(x_v, y(x_v); h) = y(x_v) - y(x_v - h) - h\sum_{j=0}^{p-1} b_{pj} f(x_v - jh, y(x_v - jh))$$

respectively.

For preparations of the succeeding discussion, we give two lemmas.

Lemma 1 ([1]) *For the Adams-Bashforth-Moulton pair of order p in the $P(EC)^m E$ mode, the identity*

$$\sum_{j=0}^{p-1}[\gamma_j \nabla^j f_{v-1}^{[m]} - \hat{\gamma}_j \nabla^j f_v^{[m]}] = -\gamma_{p-1}\nabla^p f_v^{[m]}$$

holds and, for the exact solution $y(x)$, the identity

$$T_{p2n} - T_{p1n} = -\gamma_{p-1} h \nabla^p y'(x_v)$$

also holds.

The following lemma plays an important role in the proof of Theorem 3.

Lemma 2 ([1]) *For any polynomial $P_i(x)$ of degree i, the equality*

$$\sum_{j=0}^{p}(-1)^j P_i(j)\begin{pmatrix} p \\ j \end{pmatrix} = 0 \quad for \quad 0 \le i \le p-1$$

holds.

3. Order of errors in the i-th correction

For the convenience of the later discussion, let us suppose that

$$y_\mu = y_\mu^{[i]} \quad (\mu = 0, 1, \ldots, p-1;\ i = 0, 1, \ldots, m)$$

and put

$$e_n^{[i]} = y_n^{[i]} - y(x_n) \quad (n = 0, 1, \ldots, N;\ i = 0, 1, \ldots, m).$$

Then we have the following theorem.

Theorem 1 *In the $P(EC)^m E$ mode, under the assumptions in Section 2, for a suitably chosen h, there exists a positive constant K such that*

$$|e_n^{[i]}| \leq Kh^p \ (n = 0, 1, \ldots, N;\ i = 0, 1, \ldots, m). \tag{4}$$

For the proof of this theorem refer to [4].

4. Asymptotic formula

For the asymptotic formula of $e_n^{[m]}$, we have the following theorem.

Theorem 2 *In the $P(EC)^m E$ mode, under the assumptions in Section 2, the relation*

$$e_n^{[m]} = h^p e(x_n) + O(h^{p+1}) \quad (n = 0, 1, \ldots)$$

holds. Here $e(x)$ is the magnified error function, which is the solution of the differential equation

$$e' = g(x)e - Cy^{(p+1)}(x), \quad e(x_0) = 0,$$

where C is the error constant.

For the proof of this theorem refer to [4].

5. Behavior of $y_v^{[0]} - y_v^{[m]}$

Put

$$w_j^{[i]} = \int_0^1 f_y(x_j, y(x_j) + \theta e_j^{[i]})\, d\theta\,,$$

$$\gamma_j(p,q) = \int_0^1 f_y(x_j, y_j^{[p]} + \theta(y_j^{[q]} - y_j^{[p]}))\, d\theta\,,$$

$$S_j(k) = \prod_{i=0}^{k-1} \gamma_j(i, i+1)\,.$$

When $m \geq 1$, for $0 \leq j \leq p-1$, suppose that $y_j = y_j^{[m]}$, $f_j = f_j^{[m]}$, $w_j = w_j^{[m]}$ and $e_j = e_j^{[m]}$. Put

$$\Lambda_z = (b_{p0})^{m+1} \gamma_z(0,m) S_z(m)$$

and

$$U_z = (b_{p0})^m S_z(m)\{1 - h b_{p0} \gamma_z(0,m)\}\,,$$

then some manipulations, we have

$$\begin{aligned} (5) \qquad e_z^{[m]} &= e_{z-1}^{[m]} + h \sum_{j=0}^{p-1} b_{pj} w_{z-j}^{[m]} e_{z-j}^{[m]} - T_{p2,z-p+1} \\ &\quad - \{h^{m+1} U_z/(1 - h^{m+1}\Lambda_z)\} \\ &\quad \times \{\sum_{j=0}^{p-1} b_{pj} w_{z-j}^{[m]} e_{z-j}^{[m]} - \sum_{j=1}^{p} a_{pj} w_{z-j}^{[m]} e_{z-j}^{[m]}\} \\ &\quad - \{h^m U_z/(1 - h^{m+1}\Lambda_z)\}(T_{p1,z-p+1} - T_{p2,z-p+1})\,. \end{aligned}$$

Here we give the following proposition.

Proposition 1 *For $z \geq p$ the relation*

$$h b_{p0}(f_z^{[m]} - f_z^{[m-1]}) = O(h^{p+m+1})$$

holds.

Proof. Since

$$(f_z^{[m]} - f_z^{[m-1]}) = \gamma_z(m-1,m)(e_z^{[m]} - e_z^{[m-1]}),$$

by some manipulations and Lemma 1, it follows that

$$\begin{aligned}
hb_{p0}(f_z^{[m]} - f_z^{[m-1]}) &= (hb_{p0})^m S_z(m)(e_z^{[1]} - e_z^{[0]}) \\
&= (hb_{p0})^m S_z(m)\{1 - hb_{p0}\gamma_z(0,m)\} \\
&\quad \times (h\gamma_{p-1}\nabla^p w_z^{[m]} e_z^{[m]} + T_{p1,z-p+1} - T_{p2,z-p+1}) \\
&\quad /\{1 - (hb_{p0})^{m+1}\gamma_z(0,m)S_z(m)\} \\
&= O(h^{p+m+1}).
\end{aligned}$$

This completes the proof.

For $y_v^{[0]} - y_v^{[m]}$, we have the following theorem.

Theorem 3 *In the $P(EC)^mE$ mode, under the assumptions in Section 2, the relation*

$$y_v^{[0]} - y_v^{[m]} = T_{p2n} - T_{p1n} + \varepsilon_{p\rho v} + O(h^{2p+1})$$

holds, where $\rho = \min(m,p)$ and

$$\varepsilon_{p\rho v} = O(h^{p+i+1}) \quad for \quad v \geq i(p-1)+1; 1 \leq i \leq \rho.$$

Proof. From Eqs.(2) and(3), by Lemma 1, we obtain

$$\begin{aligned}
(6)\quad y_v^{[0]} - y_v^{[m]} &= -h\gamma_{p-1}\nabla^p\{f_v^{[m]} - f(x_v, y(x_v))\} \\
&\quad - h\gamma_{p-1}\nabla^p\{f(x_v, y(x_v))\} - hb_{p0}(f_v^{[m]} - f_v^{[m-1]}) \\
&= -h\gamma_{p-1}\nabla^p w_v^{[m]} e_v^{[m]} + T_{p2n} - T_{p1n} \\
&\quad - hb_{p0}(f_v^{[m]} - f_v^{[m-1]})
\end{aligned}$$

For $v \geq 0 \quad k \geq 0$, we have

$$\begin{aligned}
e_{v+k}^{[m]} &= h^p e(x_v) + O(h^{p+1}) \\
&= P_0(k) + O(h^{p+1}).
\end{aligned}$$

From Eq.(5) and Lemma 2, we see that

$$e_{v+k}^{[m]} = e_v^{[m]} + h\sum_{\ell=1}^{k}\sum_{j=0}^{p-1} b_{pj} w_{v+\ell-j}^{[m]} e_{v+\ell-j}^{[m]} - \sum_{\ell=1}^{k} T_{p2,v+\ell-p+1}$$

$$
\begin{aligned}
& -\sum_{\ell=1}^{k} h^m U_{v+\ell}/(1-h^{m+1}\Lambda_{v+\ell})(h\gamma_{p-1}\nabla^p w_{v+\ell}^{[m]} e_{v+\ell}^{[m]} \\
& + T_{p1,v+\ell-p+1} - T_{p2,v+\ell-p+1}) \\
= \ & e_v^{[m]} + h\sum_{\ell=1}^{k}\sum_{j=0}^{p-1} b_{pj} g_{v+\ell-j} e_{v+\ell-j}^{[m]} - \sum_{\ell=1}^{k} T_{p2,v+\ell-p+1} \\
& -\sum_{\ell=1}^{k}(hb_{p0})^m(1-hb_{p0}g_{v+\ell})/\{1-(hb_{p0}g_{v+\ell})^{m+1})\} \\
& \times (h\gamma_{p-1}\nabla^p g_{v+\ell} e_{v+\ell}^{[m]} + T_{p1,v+\ell-p+1} - T_{p2,v+\ell-p+1}) \\
& + O(h^{2p+1}) + O(h^{2p+m+1}).
\end{aligned}
$$

For $v \geq p$, we obtain

$$
\begin{aligned}
e_{v+k}^{[m]} = \ & e_v^{[m]} + khg_{v+1}e_{v+1}^{[m]} - kT_{p2,v-p+2} \\
& + k(hb_{p0})^m(T_{p1,v-p+2} - T_{p2,v-p+2}) \\
& + O(h^{p+2}) + O(h^{p+m+2}) + O(h^{2p+1}) \quad (p \geq 1).
\end{aligned}
$$

For $v \geq 2(p-1)+1$, put $P_1(k) = \alpha_0 + \alpha_1 k$, it follows that

$$
\begin{aligned}
& \sum_{\ell=1}^{k}\sum_{j=0}^{p-1} b_{pj}\{g_v + (\ell-j)hg_v' + O(h^2)\}\{P_1(\ell-j) + O(h^{p+2})\} \\
= \ & \sum_{\ell=1}^{k}\sum_{j=0}^{p-1} b_{pj}\{g_v\alpha_0 + (\ell-j)(\alpha_0 hg_v' + g_v\alpha_1) + O(h^{p+2})\} \\
= \ & \sum_{\ell=1}^{k}\{g_v\alpha_0 + (\ell-\frac{1}{2})(\alpha_0 hg_v' + \alpha_1) + O(h^{p+2})\} \\
= \ & kg_v\alpha_0 + \{\frac{k(k+1)}{2} - \frac{k}{2}\}(\alpha_0 hg_v' + g_v\alpha_1) + O(h^{p+2}),
\end{aligned}
$$

$$
\sum_{\ell=1}^{k} T_{pi,v+\ell-p+1} = kT_{pi,v-p+1} + \frac{k(k+1)}{2}hT'_{pi,v-p+1} + O(h^{p+3}) \quad (i=1,2),
$$

where $T'_{pi,v-p+1} = T'_{pi}(x_v, y(x_v); h) \quad (i=1,2)$,

$$
\sum_{\ell=1}^{k}(hb_{p0}g_{v+\ell})^m(1-hb_{p0}g_{v+\ell})(h\gamma_{p-1}\nabla^p g_{v+\ell}e_{v+\ell}^{[m]})/\{1-(hb_{p0}g_{v+\ell})^{m+1}\}
$$

$$= O(h^{p+m+3})$$

and

$$\begin{aligned}
&\sum_{\ell=1}^{k}(hb_{p0}g_{v+\ell})^m(1-hb_{p0}g_{v+\ell})/\{1-(hb_{p0}g_{v+\ell})^{m+1}\}\\
&\qquad \times (T_{p1,v+\ell-p+1}-T_{p2,v+\ell-p+1})\\
&= \sum_{\ell=1}^{k}[\{(hb_{p0}g_v)^m-(hb_{p0}g_v)^{m+1}+(hb_{p0})^m\ell hmg_v{}'g_v^{m-1}\}\\
&\qquad \times (T_{p1,v-p+1}-T_{p2,v-p+1})\\
&\qquad + (hb_{p0}g_v)^m\ell h(T'_{p1,v-p+1}-T'_{p2,v-p+1})+O(h^{p+m+3})]\\
&= [k\{(hb_{p0}g_v)^m-(hb_{p0}g_v)^{m+1}\}+\frac{k(k+1)}{2}(hb_{p0})^m hmg_v{}'g_v^{m-1}]\\
&\qquad \times (T_{p1,v-p+1}-T_{p2,v-p+1})\\
&\qquad + \frac{k(k+1)}{2}h(hb_{p0}g_v)^m(T'_{p1,v-p+1}-T'_{p2,v-p+1})+O(h^{p+m+3}).
\end{aligned}$$

Hence we see that

$$e_{v+k}^{[m]} = P_2(k)+O(h^{p+3}) \qquad (p \geq 2),$$

where $P_2(k)$ is a polynomial of k degree at most 2. By induction on j, if $v \geq j(p-1)+1$, we can show that

$$e_{v+k}^{[m]} = P_j(k)+O(h^{p+j+1}) \qquad (p \geq j),$$

where $P_j(k)$ is a polynomial of k degree at most j. For $v \geq 0$, we see that

$$e_{v+k}^{[m]} = h^p e(x_v)+O(h^{p+1}) = P_0(k)+O(h^{p+1}).$$

hence For $v \geq p$, by Lemma 2, it is seen that

$$\nabla^p g_v e_v^{[m]} = \sum_{j=0}^{p}(-1)^j \binom{p}{j} g_{v-j}e_{v-j}^{[m]}$$

$$
\begin{aligned}
&= \sum_{j=0}^{p}(-1)^j \binom{p}{j} \{g_v + O(h)\}\{h^p e(x_v) + O(h^{p+1})\} \\
&= \sum_{j=0}^{p}(-1)^j \binom{p}{j} \{g_v h^p e(x_v) + O(h^{p+1})\} \\
&= O(h^{p+1}).
\end{aligned}
$$

Therefore, from Eq.(6) and Proposition 1, we have

$$y_v^{[0]} - y_v^{[m]} = T_{p2n} - T_{p1n} + O(h^{p+2}) + O(h^{p+m+1}) + O(h^{2p+1}) \quad (p \geq 1).$$

For $v \geq i(p-1)+1$, it follows that

$$
\begin{aligned}
\nabla^p g_v e_v^{[m]} &= \sum_{j=0}^{p}(-1)^j \binom{p}{j} [g_v - jhg_v' + \cdots \\
&\quad + (-1)^{i-1}\{(jh)^{i-1}/(i-1)!\}g_v^{(i-1)} + O(h^i)][P_{i-1}(-j) + O(h^{p+i})] \\
&= O(h^{p+i}).
\end{aligned}
$$

Hence, we have

$$y_v^{[0]} - y_v^{[m]} = T_{p2n} - T_{p1n} + O(h^{p+i+1}) + O(h^{p+m+1}) + O(h^{2p+1}) \quad (p \geq i).$$

Therefore the proof is completed.

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